PHYSICAL REVIEW D VOLUME 11, NUMBER 3 1 FEBRUARY 1975

# Updated evaluation of the $\rho\pi\pi$ Regge residue function

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(Received 10 September 1974)

The $\rho\pi\pi$ Regge residue function is extracted from the recent $\pi\pi$ data of Hyams *et al.* and Durusoy *et al.*, for $-1.0 \leq t \leq 0.1$ GeV$^2$, with uncertainties of the order of 15% near $t=0$. Comparison is made with an earlier result based on the data of Carroll *et al.* Finally, a discussion is given of the content of a sum rule used in the analysis.

## I. INTRODUCTION AND SUMMARY

In a recent article[1] (henceforth called T1), a rigorous sum rule was derived and applied to the $\pi\pi$ data of Carroll *et al.*[2] to yield the $\rho\pi\pi$ Regge residue function for $-32\mu^2 \leq t \leq 4\mu^2$ (where $\mu$ denotes the pion mass). The data were limited in energy, however, to $M_{\pi\pi} \leq 1.48$ GeV, and were also subject to appreciable uncertainties. It is therefore desirable to apply a similar analysis to the very recent data of Hyams *et al.*[3] and Durusoy *et al.*,[4] which extend up to $M_{\pi\pi} = 1.9$ GeV. I do so here, and again find strong evidence for Regge behavior of the $\pi\pi$ charge-exchange amplitude above 1 GeV. The $\rho\pi\pi$ residue function is extracted, with uncertainties of the order of 15% near $t=0$. Again a zero is found in the residue, the present position being $t_z = -0.42$ GeV$^2$. The analysis proceeds as follows.

## II. SUM RULE

My notation and conventions will be those of T1. In brief, I denote the $\pi\pi$ elastic amplitude with isospin $I$ in the $s$ (direct) channel by $A^I(s,t)$, and the amplitude with isospin $I$ in the $t$ channel by $T^I(s,t)$. According to standard assumptions of analyticity and crossing symmetry, the $A^I$ and $T^I$ are related by

$$T^I(s,t) = \sum_{I'=0}^{2} C_{II'} A^{I'}(s,t),$$

where $C = C^{-1}$ denotes the $s$-$t$ crossing matrix. The elements we shall need here are $C_{1I} = \frac{1}{3}, \frac{1}{2}$, and $-\frac{5}{6}$ for $I = 0$, 1, and 2, respectively.

By equating fixed-$s$ and fixed-$t$ dispersion relations for $A^1(s,t)$, the following sum rule was derived in T1:

$$\int_{4\mu^2}^{\infty} \frac{ds'}{(s'-s)(s'-u)} \left[ \operatorname{Im}T^1(s',s) - \operatorname{Im}T^1(s',t) + \frac{(t-s)(2s'+s-4\mu^2)}{(s'+2s-4\mu^2)(s'-t)} \operatorname{Im}A^1(s',s) \right] = 0, \quad (1)$$

where $u$ is to be regarded as a dependent variable defined implicitly by

$$s + t + u = 4\mu^2.$$

Equation (1) is valid for real $s$ ($\pm i\epsilon$) and real $t$ ($\pm i\epsilon$) when $-32\mu^2 \leq s \leq 4\mu^2$ and, simultaneously, $-32\mu^2 \leq t \leq 4\mu^2$.

I make the standard Regge assumption that for large, positive $s$

$$\operatorname{Im}T^1(s,t) = \gamma_\rho(t)(s/\bar{s})^{\alpha_\rho(t)}, \quad (2)$$

where $\gamma_\rho$ is related by a well-known factor to the residue of the $\rho$ pole in the $J$ plane, and $\alpha_\rho$ denotes the $\rho$ trajectory. I shall use $\bar{s} = 1$ GeV$^2$, which defines the scale of $\gamma_\rho$. As in T1, I assume that

$$\alpha_\rho(t) = 0.50 + 0.90(t/\bar{s}). \quad (3)$$

Equations (1) and (2) imply that

$$\gamma_\rho(t) = f^{-1}(s,t;t;\Lambda) \left[ f(s,t;s;\Lambda)\gamma_\rho(s) + h(s,t) + \mathrm{P}\int_{4\mu^2}^{\Lambda} ds' \frac{\operatorname{Im}T^1(s',s) - \operatorname{Im}T^1(s',t)}{(s'-s)(s'-u)} \right], \quad (4)$$

where

$$f(s,t;x;\Lambda) \equiv \mathrm{P}\int_{\Lambda}^{\infty} ds' \frac{(s'/\bar{s})^{\alpha_\rho(x)}}{(s'-s)(s'-u)},$$

$$h(s,t) \equiv (t-s)\mathrm{P}\int_{4\mu^2}^{\infty} ds' \frac{(2s'+s-4\mu^2)\operatorname{Im}A^1(s',s)}{(s'-s)(s'-t)(s'-u)(s'+2s-4\mu^2)},$$

and $\Lambda$ may take on any positive value large enough for Eq. (2) to hold for $T^1(s', s)$ and $T^1(s', t)$ when $s' \geq \Lambda$. Since Eq. (2) can hold only for $\theta \leq 90°$, the minimum suitable value for $\Lambda$ must satisfy the inequalities

$$\Lambda \geq 4\mu^2 - 2s,$$

$$\Lambda \geq 4\mu^2 - 2t.$$

The virtue of Eq. (4) is that if $\gamma_\rho$ is known for any *single* value of its argument, this value of the argument can be substituted for $s$ on the right-hand side of Eq. (4), and then $\gamma_\rho(t)$ can be computed over the interval $-32\mu^2 \leq t \leq 4\mu^2$ from a knowledge of $\mathrm{Im}T^1$ between threshold and $\Lambda$, together with knowledge of the rapidly convergent integral $h(s,t)$. [In effect, the sum rule (1) fixes the *derivative* of $\gamma_\rho$, and a knowledge of the integration constant yields $\gamma_\rho(t)$.]

## III. ANALYSIS OF DATA FOR $\mathrm{Im}T^1$

I have assumed that $T^1(s,t)$ is given by the $S$, $P$, $D$, and $F$ waves of Hyams *et al.*[3] (for $I_s = 0$ and 1) and those of Durusoy *et al.*[4] (for $I_s = 2$), over the energy range of their data. The result for $\mathrm{Im}T^1(s,t)$ is presented in Fig. 1 as a function of $s$, for six different values of $t$. Noting the different vertical scales in Fig. 1, we see that $\mathrm{Im}T^1$ displays a very definite zero near $t = -0.4$ GeV$^2$. I interpret this to mean that $\gamma_\rho(t)$ vanishes near $-0.4$ GeV$^2$.

We can extract $\gamma_\rho(t)$ directly from the data by taking an average

$$\overline{\gamma}_\rho(t) = [s_b - s_a(t)]^{-1} \int_{s_a(t)}^{s_b} ds (s/\overline{s})^{-\alpha_\rho(t)} \mathrm{Im}T^1(s,t). \tag{5}$$

The lower limit of integration $s_a$ must depend on $t$ because Eq. (2) can hold only for $\theta \leq 90°$, i.e., only when $s \geq (4\mu^2 - 2t)$. In T1, I took $s_a(t)$ to be the greater of 1.0 GeV$^2$ and $(4\mu^2 - 2t)$. The data lie in the resonance region, however, where Regge theory can hold only in the sense of local averages. In this work, I therefore choose $s_a$ and $s_b$ to lie midway between resonant values of $s$, the relevant ones being $s_\rho = 0.59$ GeV$^2$, $s_f = 1.61$ GeV$^2$, $s_g = 2.84$ GeV$^2$, and $s(f_0 \text{ rec.}) = 4.0$ GeV$^2$, where the latter refers to the presumed $J = 4$ recurrence of the $f_0$ resonance.

It is convenient to define

$$s_1 \equiv \tfrac{1}{2}(s_\rho + s_f),$$

$$s_2 \equiv \tfrac{1}{2}(s_f + s_g),$$

$$s_3 \equiv \tfrac{1}{2}[s_g + s(f_0 \text{ rec.})].$$

When $s_1 \geq (4\mu^2 - 2t)$, I use $s_a = s_1$. When $s_1 < (4\mu^2 - 2t)$, I use $s_a = s_2$. In both cases, I use $s_b = s_3$. The resulting value for $\overline{\gamma}_\rho(t)$ is shown in Fig. 2(a). In principle, $\overline{\gamma}_\rho(t)$ is discontinuous at $t = -0.51$ GeV$^2$, where $s_a$ changes from $s_1$ to $s_2$. In practice, however, the discontinuity is not visible on the scale of Fig. 2. From this result for $\overline{\gamma}_\rho$, I conclude that $\gamma_\rho$ vanishes at

$$t_z = -0.42 \text{ GeV}^2. \tag{6}$$

I have set $s = t_z$ on the right-hand side of Eq. (4), and computed $\gamma_\rho(t)$ for $-32\mu^2 \leq t \leq 4\mu^2$, for a range of $\Lambda$ over the interval $s_a(t) \leq \Lambda \leq s_b$. As in T1, $\gamma_\rho(t)$ is quite insensitive (~10% variations) to these changes in $\Lambda$. The result for $\gamma_\rho(t)$, averaged over $\Lambda$, is shown in Fig. 2(a). Again the discontinuity at $t = -0.51$ GeV$^2$ (due to the change in $s_a$ there) is not visible on the scale of the figure.

Observe that the results of Eqs. (4) and (5) are *indistinguishable* on the scale of Fig. 2, except

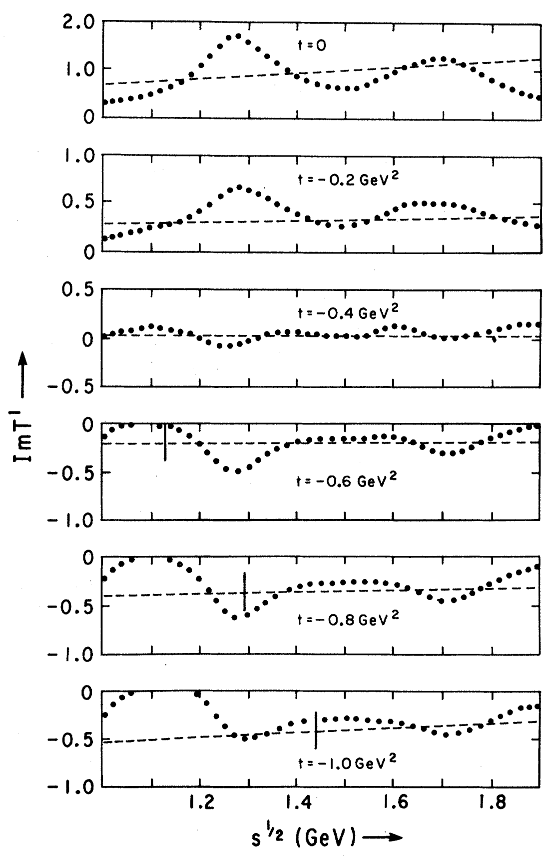


FIG. 1. Dotted curves depict $\mathrm{Im}T^1(s,t)$ implied by $S$, $P$, $D$, and $F$ waves of Hyams *et al.* (for $I_s = 0$ and 1) and Durusoy *et al.* (for $I_s = 2$). Dashed curves depict $\mathrm{Im}T^1$ implied by Eqs. (2), (3), and (7). For $t \leq -0.6$ GeV$^2$, vertical bars denote energies above which $\theta \leq 90°$, hence energies above which dotted and dashed curves are expected to agree.

near the end points of the interval where Eq. (4) is valid. This very close agreement provides strong support for my working hypothesis that Eq. (2) is valid, in the sense of local averages, for $s > 1$ GeV$^2$.

For the sake of future applications, I note that $\bar{\gamma}_\rho(t)$ is given within ±0.01 over the interval $-1.0 \leq t \leq 0.1$ GeV$^2$ by

$$\bar{\gamma}_\rho(t) = 0.67 + 1.78(t/\bar{s}) + 0.41(t/\bar{s})^2 - 0.17(t/\bar{s})^3. \quad (7)$$

Equation (7) represents my preferred result for $\gamma_\rho$ in this work, since it agrees with the sum rule for $-32\mu^2 < t < 4\mu^2$ but is valid over the wider range of $t$ cited above.

## IV. ANALYSIS OF Re$T^1$

As a further test of Regge behavior, I also analyze Re$T^1$. It is convenient to express the Regge prediction for Re$T^1$ in a form analogous to Eq. (2), i.e.,

$$\mathrm{Re}T^1(s,t) = \beta_\rho(t)(s/\bar{s})^{\alpha_\rho(t)}, \quad (8)$$

where Regge theory implies that

$$\beta_\rho(t) = \left[\frac{1-\cos\pi\alpha_\rho(t)}{\sin\pi\alpha_\rho(t)}\right]\gamma_\rho(t). \quad (9)$$

In Fig. 2(b), I present the result for $\beta_\rho$ obtained from Eq. (9), using Eq. (7) for $\gamma_\rho$. I also present results for $\bar{\beta}_\rho$ defined by

$$\bar{\beta}_\rho(t) \equiv [s_b - s_a(t)]^{-1}\int_{s_a(t)}^{s_b} ds\,(s/\bar{s})^{-\alpha_\rho(t)}\,\mathrm{Re}T^1(s,t). \quad (10)$$

I find that $\bar{\beta}_\rho$ depends appreciably on whether I include the data above 1.5 GeV. The agreement between $\bar{\beta}_\rho$ and Eq. (9) is fairly good when I evaluate $\bar{\beta}_\rho$ over the interval $s_1 \leq s \leq s_2$, but is less good near $t=0$ when I use the interval $s_2 \leq s \leq s_3$. The reason may be that for $s > s_2$ the *real* parts of partial waves with $l \geq 4$ make significant contributions to Re$T^1$, which have been neglected here. One expects on quite general grounds that the real parts of high-order partial waves will become appreciable at lower energies than the imaginary parts, and the expected $J=4$ recurrence of the $f_0$ resonance should make Im$A^{(4)0}$ large near $M_{\pi\pi} = 2$ GeV. Thus it would seem unwise to neglect Re$A^{(4)0}$ in the $g$ region and above, and perhaps Re$A^{(4)2}$.

It is worth remarking that resonances can build up a Regge-behaved Im$T^1$ at fairly low energies, whereas unitarity impedes an early onset of Regge behavior for Re$T^1$. In particular, the local average of Re$A^{(l)I}$ *vanishes* in the neighborhood of an elastic resonance, so the burden of Regge behavior for Re$T^1$ falls on the nonresonant amplitude $A^2$ at low energies. Unitarity places a bound on the real part of each partial wave, however, and the centrifugal barrier suppresses high-order partial waves at low energies. This combination of factors leads one to expect Regge behavior to set in at a somewhat higher energy for Re$T^1$ than for Im$T^1$. (In the single-term Veneziano model for $\pi\pi$ scattering, both Im$T^1$ and Re$T^1$ have Regge behavior above 1 GeV but the latter requires Re$A^{(0)2}$ to exceed the unitarity bound of $-\frac{1}{2}$, reaching $-0.8$ between 1.0 and 1.5 GeV.) In light of the preceding remarks, I regard the fairly good agreement between solid curve and open circles in Fig. 2(b) as further support for my hypothesis that Eq. (2) is valid above 1 GeV.

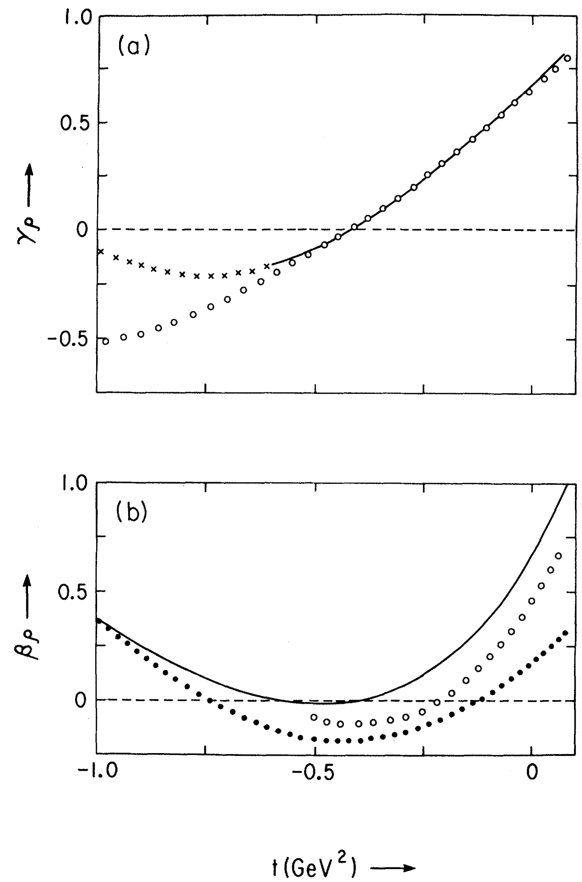


FIG. 2. (a) Open circles depict $\bar{\gamma}_\rho$ defined by Eq. (5). Solid curve displays $\gamma_\rho$ computed from Eq. (4). Crosses depict result of Eq. (4) in forbidden region $t < -32\mu^2$. (b) Solid curve displays $\beta_\rho$ implied by Eqs. (7) and (9). Open circles depict $\bar{\beta}_\rho$ defined by Eq. (10), with $s_a = s_1$ and $s_b = s_2$ (the curve terminates where $4\mu^2 - 2t = s_1$, for reasons discussed in text). Closed circles depict $\bar{\beta}_\rho$ with $s_a = s_2$ and $s_b = s_3$.

## V. UNCERTAINTIES IN $\gamma_\rho(t)$

The uncertainties in $\gamma_\rho(t)$ are of two kinds: statistical and systematic. The statistical uncertainties are quite small, as may be seen in the following way.

The quantities extracted directly from the data [if one assumes Eq. (3) for $\alpha_\rho(t)$] are $\overline{\gamma}_\rho(t)$ and $\overline{\beta}_\rho(t)$. For any *single* pair of values for $s$ and $t$, the statistical (experimental) uncertainty in $T^1(s,t)$ is appreciable. Equation (5), however, expresses $\overline{\gamma}_\rho(t)$ as an integral of $\mathrm{Im}T^1(s,t)$ over a wide range of $s$. If data at $N$ different values for $s$ are included in the integral, then the resulting statistical uncertainty in $\overline{\gamma}_\rho(t)$ is roughly proportional to $N^{-1/2}$. $\mathrm{Im}T^1$ is primarily determined by $\mathrm{Im}A^0$ and $\mathrm{Im}A^1$, and Hyams *et al.* report phase shifts and inelasticities at 37 different values for $s$ between $s_1$ and $s_3$. Hence the statistical uncertainty in $\overline{\gamma}_\rho(t)$ is only about $\frac{1}{6}$ of that in any single value for $\mathrm{Im}T^1(s,t)$. This makes the *statistical* uncertainty in $\overline{\gamma}_\rho(t)$ extremely small (Ref. 3 is a fairly high-statistics experiment). Similar remarks apply to the $\gamma_\rho$ of Eq. (4).

The primary uncertainty in $\gamma_\rho(t)$ is systematic, and therefore difficult to estimate with precision. Apart from possible systematic errors in the data, I have assumed that $\mathrm{Im}T^1$ has Regge behavior above 1 GeV. It is conceivable that the apparent consistency between this assumption and the data is misleading, and that $\mathrm{Im}T^1$ simply is not dominated by $\rho$ exchange anywhere below 2 GeV. If this were so, then my result for $\gamma_\rho$ would need bear no resemblance to the $\rho\pi\pi$ Regge residue function.

Although I have taken my preferred result for $\gamma_\rho$ directly from the data via Eq. (5), Eq. (4) is still quite useful, because it depends *strongly* on the assumption that $\mathrm{Im}T^1$ has Regge behavior for all $s>\Lambda$. It is the extreme consistency between the results of Eqs. (4) and (5), as shown in Fig. 2(a), which gives me confidence that $\mathrm{Im}T^1$ has Regge behavior above 1 GeV.

If I took the maximum discrepancy between the two curves in Fig. 2(a) as a measure of the uncertainty, then I would say that $\gamma_\rho(t)$ is known within $\pm0.05$ for $-0.6\leq t\leq 0.1$ GeV$^2$. This estimate of uncertainty would not be entirely reliable, however, and I prefer to make the more conservative statement that if Eq. (3) is correct then $\gamma_\rho(t)$ is probably given within $\pm0.10$ by Eq. (7). This amounts to an uncertainty of $\pm15\%$ near $t=0$.

The value obtained for $\gamma_\rho$ depends of course on the value assumed for $\alpha_\rho$. If I had assumed, for example, that

$$\alpha_\rho(t)=0.60+1.00(t/\overline{s}), \qquad (11)$$

then I would have obtained

$$\gamma_\rho(t)\cong 0.62+1.60(t/\overline{s})+0.23(t/\overline{s})^2-0.22(t/\overline{s})^3. \qquad (12)$$

It is scarcely possible to tell from the present data whether Eq. (11) for $\alpha_\rho$ is really better or worse than Eq. (3), because Eqs. (11) and (12) lead to the same $\mathrm{Im}T^1$ as Eqs. (3) and (7), within an average discrepancy of only $\pm4\%$ for $s_1\leq s\leq s_3$, $-1.0\leq t\leq 0.1$ GeV$^2$. The discrepancy is greater ($\pm7\%$) near $t=0$, but this is a region where the experimental $\mathrm{Im}T^1$ oscillates with a large amplitude as $s$ varies (see Fig. 1). It is difficult to determine $\alpha_\rho(0)$ from an interval of $s$ containing only two oscillations ($f_0$ and $g$). For any preferred choice of $\alpha_\rho(t)$, one may obtain the corresponding $\gamma_\rho$ by assuming a cubic form for $\gamma_\rho(t)$, with coefficients to be determined from a least-squares fit to the $\mathrm{Im}T^1(s,t)$ implied by Eqs. (3) and (7).

## VI. THE DATA OF CARROLL *et al.*

The results (6) and (7) for $\gamma_\rho$ are close to those obtained in T1 from the data of Carroll *et al.*[2] However, the position of $t_g$ has increased by 0.1 GeV$^2$, and $\gamma_\rho(0)$ has decreased by 0.15.

Both of the aforementioned changes are due to a discrepancy in the $I_s=2$ $D$-wave phase shift $\delta_2^2$. More specifically, Carroll *et al.* reported a $\delta_2^2$ of $-20°$ in the $f_0$ region. This value is very much larger than that reported by Durusoy *et al.*,[4] whose work confirms the result of many earlier authors[5] that $-5°\leq\delta_2^2\leq 0$ for $M_{\pi\pi}\leq 1.5$ GeV. A relatively large value for $\delta_2^2$ affects $T^1$ appreciably, because $|C_{12}|=\frac{5}{2}|C_{10}|$, while of course $(2l+1)=5$. A simple analysis reveals that this difference in the value used for $\delta_2^2$ is largely responsible for the slight differences between my present results for $\gamma_\rho$ and those reported in T1. Since the smaller value for $\delta_2^2$ used here has been confirmed by many authors,[5] the $\gamma_\rho$ presented here supersedes that reported in T1. [I remark also that the agreement between the results of Eqs. (4) and (5) is better here than in T1.]

## VII. FURTHER CONTENT OF SUM RULE

If one interchanges $s$ with $t$ in Eqs. (1) and (4), one obtains superficially independent equations. By using these new equations, one can eliminate $\gamma_\rho(s)$ from the right-hand side of Eq. (4), with the result

$$\gamma_\rho(t) = [f(t,s;s;\Lambda)f(s,t;t;\Lambda) - f(s,t;s;\Lambda)f(t,s;t;\Lambda)]^{-1}$$
$$\times\left\{f(t,s;s;\Lambda)\left[h(s,t) + \mathrm{P}\int_{4\mu^2}^{\Lambda} ds' \,\frac{\mathrm{Im}T^1(s',s) - \mathrm{Im}T^1(s',t)}{(s'-s)(s'-u)}\right]\right.$$
$$\left. + f(s,t;s;\Lambda)\left[h(t,s) + \mathrm{P}\int_{4\mu^2}^{\Lambda} ds' \,\frac{\mathrm{Im}T^1(s',t) - \mathrm{Im}T^1(s',s)}{(s'-t)(s'-u)}\right]\right\}. \qquad (13)$$

The right-hand side of Eq. (13) is completely determined by integrals over absorptive parts (together with the known function $f$). If one computes $\gamma_\rho(t)$ from Eq. (13), however, one finds the result to be *extremely* sensitive to the $f_0$ and $g$ resonance parameters. Hence it is difficult to obtain a reliable result for $\gamma_\rho$ in this way.

The left-hand side of Eq. (13) is manifestly independent of $s$, so the right-hand side must be independent of the value chosen for $s$. Using the experimental values for $\rho$ resonance parameters, however, I find that the right-hand side depends *strongly* on $s$, *unless* I use values for the $f_0$ and $g$ parameters very near the experimental ones. Thus Eq. (13) seems to imply an interesting relation between the $\rho$, $f_0$, and $g$ resonance parameters. This possibility may merit further study.

[1]E. P. Tryon, Phys. Rev. D 8, 1586 (1973).

[2]J. T. Carroll *et al.*, Phys. Rev. Lett. 28, 318 (1972).

[3]B. Hyams *et al.*, Nucl. Phys. B64, 134 (1973). I use the energy-dependent amplitudes. I remark that a typesetting error is present in Eq. (13b), where $\gamma_\pi{}^2$ is dimensionally incorrect and does not reproduce the phase shift $\delta_2^0$ of Fig. 5.

[4]N. B. Durusoy *et al.*, Phys. Lett. 45B, 517 (1973).

[5]Cf. W. Hoogland *et al.*, Nucl. Phys. B69, 266 (1974), and references cited therein.